The Open University
Mathematics and Computing/Science/Technology
MST204 Mathematical Models and Methods

MST204

Assignment Booklet II

Contents

The marks allocated to each part of each TMA question are indicated in brackets in the margin.

Please send all your answers to each tutor-marked assignment (TMA), together with an appropriately completed assignment form (PT3), to reach your tutor on or before the appropriate cut-off date shown above. Remember to fill in the correct *Assignment Number* as listed above and allow sufficient time in the post for each assignment to reach its destination on or before the cut-off date.

Similarly, each completed computer-marked assignment (CMA) form should be posted in an envelope marked *Computer-marked Assignment* so as to arrive at the Open University by the appropriate cut-off date.

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17.1

This assignment covers *Units 9, 12, 14* and *15.*

Question 1 (*Unit 9*)

(i) By using the Gaussian elimination method, determine whether the following sets of simultaneous equations have

- no solution,
- a unique solution,
- an infinite number of solutions.

In each case, if the set of equations has a unique solution find the solution, and if it has an infinite number of solutions find the general solution.

(a) [*11*]

$$\begin{aligned} 3x_1 + 5x_2 - 6x_3 &= 4 \\ x_1 + 2x_2 + 3x_3 &= 3 \\ 3x_1 + 6x_2 + 8x_3 &= 8 \end{aligned}$$

(b) [*7*]

$$\begin{aligned} 3x_1 + 5x_2 - 6x_3 &= 4 \\ x_1 + 2x_2 + 3x_3 &= 3 \\ 3x_1 + 6x_2 + 9x_3 &= 9 \end{aligned}$$

(ii) To investigate the absolute conditioning of the set of simultaneous equations in part (i)(a), the value of the right-hand side in the third equation is changed from 8 to $8+\varepsilon$, where ε is a small number, so that the equations become

$$\begin{aligned} 3x_1 + 5x_2 - 6x_3 &= 4, \\ x_1 + 2x_2 + 3x_3 &= 3, \\ 3x_1 + 6x_2 + 8x_3 &= 8+\varepsilon. \end{aligned}$$

Find the exact solution of the modified simultaneous equations. What conclusions (if any) can you draw about the absolute conditioning of the original set of equations from these results? [*7*]

Question 2 (*Unit 12*)

This question is concerned with steady-state heat loss through an external wall of a room of a house. The wall is a cavity wall, consisting of two layers of brick with an air gap between them. The wall is 2.5 metres high and 3 metres wide. The layers of brick are each 0.11 metres thick, and the width of the air gap separating them is 0.05 metres. In the wall there is a window 1.5 metres high and 1.5 metres wide; it consists of a single pane of glass 0.006 metres thick. The thermal conductivity of brick is $0.6\,\mathrm{W\,m^{-1}\,°C^{-1}}$, and the thermal conductivity of glass is $1.0\,\mathrm{W\,m^{-1}\,°C^{-1}}$. The convective heat transfer coefficients for the surfaces of both the brick wall and the window are $10\,\mathrm{W\,m^{-2}\,°C^{-1}}$ for the inside of the house and $20\,\mathrm{W\,m^{-2}\,°C^{-1}}$ for the outside of the house. The combined heat transfer coefficient for the air gap of the cavity wall is $1.5\,\mathrm{W\,m^{-2}\,°C^{-1}}$. The outside and inside air temperatures are constant at 2 °C and 20 °C, respectively.

(i) Calculate the U value for the cavity wall, and the U value for the window. [*4*]

(ii) Find the total amount of heat lost in one day through the cavity wall and window. [*6*]

(iii) Find the temperatures on each of the *four* surfaces of the brick layers of the cavity wall. [*8*]

(iv) Suppose that the wall is to be insulated by filling the cavity with plastic foam, whose thermal conductivity is $0.034\,\mathrm{W\,m^{-1}\,°C^{-1}}$. What will be the resulting reduction in heat lost per day, assuming that all other conditions remain the same? What will be the temperatures on the surfaces of the wall inside the room and outside the house with cavity insulation in place, all other conditions being unchanged? [*7*]

Question 3 (*Unit 14*)

(i) Show that the area of a triangle ABC is equal to half of the magnitude of the cross product $\overrightarrow{AB} \times \overrightarrow{AC}$. [3]

(ii) The position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ of three particles A, B, C with respect to a fixed origin O are given by

$$\mathbf{a} = t\mathbf{i} + \mathbf{j} + (t+2)\mathbf{k},$$
$$\mathbf{b} = (2t+1)\mathbf{i} + (t+1)\mathbf{j} + (t+1)\mathbf{k},$$
$$\mathbf{c} = (t+2)\mathbf{i} + (t+4)\mathbf{j} + (2t+3)\mathbf{k},$$

where t represents time, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are fixed Cartesian unit vectors.

(a) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of t, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [3]

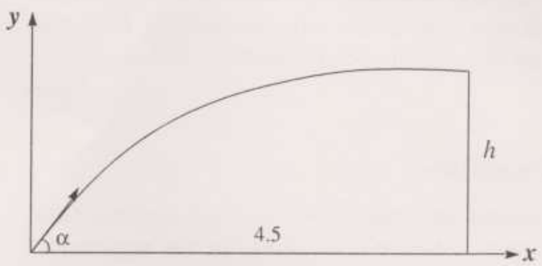
(b) Find a unit vector perpendicular to the plane containing the triangle ABC, and deduce that the particles move in such a way that successive positions of this plane are always parallel to each other. [8]

(c) Find the area of the triangle ABC. [3]

(d) Find the value of t for which the area of the triangle ABC is a minimum, and find this minimum area. [4]

(e) Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ when the area of the triangle ABC is a minimum. [4]

Question 4 (*Unit 15*)



A netball player throws the ball from a point h metres below and $4\frac{1}{2}$ metres horizontally from the net, with the purpose of making the ball fall through the net. The ball is thrown with a speed of $9\,\mathrm{m\,s^{-1}}$ at an angle of projection of α to the horizontal. Throughout the question you should neglect air resistance and model the ball as a particle. You should further assume that the acceleration due to gravity is constant and of magnitude $g = 10\,\mathrm{m\,s^{-2}}$.

(i) *From Newton's second law of motion*, show that during its flight the horizontal and upward vertical displacements x and y of the ball from its point of projection satisfy the equation

$$y = x\tan\alpha - \tfrac{5}{81}x^2\sec^2\alpha.$$ [7]

(ii) If the player throws the ball from a point $h = 1$ metre below the net, find the possible angles of projection of the ball in order that the net should be on the ball's trajectory. [6]

(iii) By examining the vertical component of the ball's velocity, show that the ball actually falls through the net from above (rather than passing through the net from below travelling upwards) for both the possible angles of projection. [6]

(iv) Show that, for the given speed of projection and horizontal distance from the net, the maximum achievable height of the net above the point of projection for different angles of projection α is 2.8 metres. [6]

[*You may find the following identity useful*:

$\sec^2\alpha = 1 + \tan^2\alpha$.]

This assignment covers *Unit M*.

There are two questions in this assignment, each marked out of 50.

The purpose of this TMA is to help you practise the modelling skills described in *Unit M*, which you will need when you come to tackle the modelling exercise for TMA 07. It consists of two questions. In the first we invite you to analyse a piece of modelling that has been carried out by a professional mathematician. We hope that this will help you to understand better the processes involved in modelling. This question is similar to those we asked in Section 2 of *Unit M*, about the drug therapy and the skid marks models. The second question is a problem for you to solve by mathematical modelling, and has a similar purpose to the exercise in Section 5 of *Unit M*. In both questions the organization of the work follows the outline of the modelling process given in the summary of Section 4. You may find it useful to glance over this to refresh your memory before you begin.

This TMA is designed to take no longer than any of TMAs 01 to 03. If you find that you are spending a lot of time on it, then you are probably reading more into the questions than was intended.

Question 1

The following passage is part of an article describing a mathematical model of the costs and benefits of insulating houses against heat loss. Read the passage, and then tackle the questions which follow it. You will find that there are several gaps in the passage; some of the questions ask you to work out what is missing. However, you should be able to follow the sense of the article despite these omissions.

Saving Heating Costs with Loft Insulation

This article is concerned with the financial benefits which may be derived from insulating the loft, or roof space (the space between the ceilings of the uppermost rooms and the tiled or slated roof itself), of one's house. This is done by laying insulating material such as fibreglass over what one might call the floor of the loft, so as to decrease the flow of heat through the ceilings of the rooms below.

Loft insulation reduces heat loss from a house and saves the householder money by cutting the cost of heating; but against this saving must be set the cost of installing the insulation itself. Many houses in the UK lack loft insulation; and of those houses that have it, many have too little. The recommended thickness for fibreglass loft insulation in the UK is 100 mm; few houses are insulated to this level. The cost of insulating one's house clearly increases if one installs thicker insulation, and this is no doubt a major factor influencing how much insulation people install; but using a thin layer of insulation may be a false economy if the extra savings on the heating bill from thicker insulation outweigh the additional costs of installation. The purpose of this article is to investigate how the overall financial benefits of installing loft insulation vary with the thickness of the insulation installed, and to find the thickness for which the financial benefit is greatest.

The cost of heating is a continuing, or recurrent, cost, whereas expenditure on insulation is a one-off, capital, cost. It is necessary to find a suitable way of combining these two rather different quantities. The method of combining a recurrent and a capital cost which will be used here is to add to the recurrent cost the interest which would have been received if the capital had been invested instead of spent. The idea is that every £1 spent on insulation, if it had been invested instead at (say) 5% per annum, would have earned the investor 5p each year; by choosing to spend that £1 on insulation, one foregoes the 5p interest, and this can be regarded as an annual cost which accounts for the outlay on the insulation.

The problem, therefore, is to find the optimum thickness of loft insulation, taking into account both the cost of heating and expenditure on the insulation. The

approach to solving this problem which will be adopted in this article is to obtain an explicit formula for $c(t)$, the overall annual cost in £ associated with insulating each square metre of the loft space with insulation t metres thick, and to find the value of t for which $c(t)$ is a minimum, using the usual methods of the calculus. One consequence of this approach is that the only factors which are taken into account are financial ones.

As has been explained above, the total cost $c(t)$ is the sum of two components, the cost of heating, and the interest foregone on the cost of installation; each of these terms will also depend on the thickness of the insulation. Thus

$$c(t) = h(t) + i(t), \tag{1}$$

where $h(t)$ is the annual cost in £ of supplying the heat lost into the roof space through one square metre of the ceiling, and $i(t)$ is the annual interest in £ received on an invested sum equal to the capital cost of installing one square metre of insulation, when the thickness of the insulation is t metres. Each of these two components will be considered separately.

To find an explicit expression for $h(t)$, it will be necessary to consider the loss of heat into the roof space of the house through the ceiling of each room immediately below it. In any particular case the rate of loss of heat will depend on the materials from which the house is constructed; but the model to be developed here will apply to any house whose ceilings adjacent to the roof are made from a single material and are of uniform thickness. For simplicity's sake, the effects of the timbers in the roof to which the ceilings are attached will be ignored; and only conduction (and not convection or radiation) will be considered. The temperature in the roof space will be taken to be equal to the outside temperature. The rate of loss of heat, q watts, through 1 square metre of ceiling, when insulation t metres thick is installed, is given by

$$\begin{aligned} q &= U(\theta_{\text{in}} - \theta_0) && (2)\\ &= \frac{\kappa}{t}(\theta_0 - \theta_{\text{out}}), && (3) \end{aligned}$$

where U is the U value for the ceiling material alone, κ is the thermal conductivity of the insulating material, and θ_{in}, θ_0 and θ_{out} are the temperatures inside the house, at the interface between the ceiling and the insulation, and outside the house, respectively. Thus

$$q = \frac{\kappa U(\theta_{\text{in}} - \theta_{\text{out}})}{\kappa + Ut}. \tag{4}$$

The heat that is being lost at this rate has to be supplied by the heating system; so the contribution of the losses into the roof space to the annual cost of heating the house is obtained by summing up the heat lost at this rate for the whole period of time that the heating system is operating. It is natural to assume that the required internal temperature θ_{in} does not change through the winter months while the heating is on. The external temperature will vary, of course; but this can be allowed for by taking for θ_{out} the average external temperature over the period in which the heating is on. Then the total heat lost into the roof space through each square metre of the ceiling during one year, Q W h, is given by

$$Q = \frac{\kappa UT\Delta\theta}{\kappa + Ut}, \tag{5}$$

where T is the total time for which the heating is running and $\Delta\theta$ is the average temperature difference between the inside and the outside of the house. (The units for the total heat lost are watt hours. One watt hour, or W h, is the energy produced by a source rated at 1 watt running for one hour. A more common unit of energy is the kilowatt hour, or kW h; one kilowatt hour is 1000 watt hours.)

Finally, the annual cost of providing the heat which is lost into the roof space through each square metre of the ceiling is obtained by multiplying Q by the cost of a watt hour of energy, say £C, which gives

$$h(t) = CQ = \frac{C\kappa UT\Delta\theta}{\kappa + Ut}. \tag{6}$$

The other component of the total cost is the cost of installation. The most economical way of insulating one's loft is to do the job oneself; it is not a very difficult job, though a dirty and tiring one, so it seems not unreasonable to suppose that this is how it is done. It is then necessary to take into account only the cost of the insulating material. This is easy to calculate because it will be proportional to the volume of material required. Suppose that the price of 1 cubic metre of insulating material is £P; then

$$i(t) = \frac{IPt}{100}, \tag{7}$$

where the rate of interest is $I\%$. Combining the two costs gives

$$c(t) = h(t) + i(t) = \frac{C\kappa UT\Delta\theta}{\kappa + Ut} + \frac{IPt}{100}. \tag{8}$$

It remains to find the value of t (if there is one) for which $c(t)$ is a minimum. Now

$$\frac{dc}{dt} = \boxed{\text{deliberate omission!}}; \tag{9}$$

thus in order for $c(t)$ to have a minimum, t must satisfy

$$(\kappa + Ut)^2 = \boxed{\text{deliberate omission!}}, \tag{10}$$

and therefore

$$t = \sqrt{\frac{100C\kappa T\Delta\theta}{IP}} - \frac{\kappa}{U}. \tag{11}$$

This gives the thickness of insulation for which the total annual cost is a minimum.

As an example, consider the following data.

- The ceilings are constructed from plasterboard which is 10 mm thick and has a thermal conductivity of $0.16\,\mathrm{W\,m^{-1}\,°C^{-1}}$.
- The insulating material to be used is fibreglass; the fact that it is usually available only in certain standard thicknesses may be ignored. A roll of fibreglass insulation 8 m long, 0.35 m wide and 100 mm thick costs £4.80. The thermal conductivity of fibreglass is $0.04\,\mathrm{W\,m^{-1}\,°C^{-1}}$.
- Interest on invested money is 5% per annum.
- Heating is required in the average house in an average year for 15 hours during the day, for 180 days during the year, and the average temperature difference between the inside and the outside of the house in that period is 8 °C.
- The running costs of the heating system are 2.5 pence per kilowatt hour.

The U value for the ceiling material alone is $16\,\mathrm{W\,m^{-2}\,°C^{-1}}$. The cost of $1\,\mathrm{m^3}$ of fibreglass is £17.14. The value of t at the minimum is therefore

$$\sqrt{\frac{100 \times 0.025 \times 0.04 \times 180 \times 15 \times 8}{1000 \times 5 \times 17.14}} - \frac{0.04}{16} \simeq 0.156,$$

and the optimal thickness is about 160 mm.

End of article

(i) *Specify the real problem*

The article is concerned with the costs of heating a house and the possible benefits of insulation – but this does not state the problem sufficiently precisely for the purposes of mathematical modelling. What is the actual problem that the article tackles? Use your own words. You should be able to give a satisfactory answer in a couple of sentences at most. [4]

(ii) *Set up a model*

(a) There are many symbols in the passage: $c, t, q, \kappa, \ldots$. Make a list of them all, explain precisely what each represents ('t is thickness' will *not* do!), give appropriate units of measurement, and say whether each represents a variable or a parameter. [6]

(b) The most important steps in the derivation of the formula for $h(t)$ (Equation 6) are those which lead to Equation 4 for the rate of loss of heat. Explain how the theory of heat conduction described in *Unit 12* leads to Equations 2 and 3. Make sure that you state any assumptions about how the transfer of heat takes place that must be made to derive these equations, whether or not they are mentioned in the article. Show in detail how Equation 4 follows from Equations 2 and 3. [*5*]

(c) In the discussion of the cost of installation, the article states that it is easy to calculate the cost of the material because it will be proportional to the volume of material required. Show how this assumption leads to Equation 7 for $i(t)$. [*2*]

(d) Many assumptions needed to derive the model are mentioned in the article, though they are not necessarily labelled as assumptions. Unfortunately, authors of articles on modelling sometimes overlook assumptions which in fact they use implicitly; and they sometimes state assumptions which are not really necessary. Each of the following statements could possibly be an assumption which is needed for the development of the model. Decide, for each, whether or not it (or something equivalent) is explicitly mentioned in the article as an assumption in deriving the model; and decide whether or not it is strictly necessary to make that assumption in order to derive the model in the way described in the article.

A The house has two stories – that is, the model would not apply to a bungalow.

B All the heat lost into the roof space of the house while heating is required is supplied by the heating system in the house.

C The insulating material used is fibreglass.

D The rate of inflation is zero during the years over which the model is supposed to apply.

State one more assumption, mentioned in the article, which is necessary for the derivation of the model, and is not among those discussed in this part of the question (part (ii)), or in part (vi)(b) below. Use your own words; do not merely copy from the article. [*5*]

(iii) *Formulate the mathematical problem*

The mathematical model obtained in the article is expressed in Equation 8, which gives the explicit formula for $c(t)$ in terms of t. What would you regard as the main mathematical *problem* in this article? Is it

- to calculate the value of t using the data given in the final paragraph?
- to solve Equation 10 for t?
- to differentiate $c(t)$ with respect to t?
- to find the value of t for which $c(t)$ is a minimum?

Or is it something else entirely? [*2*]

(iv) *Solve the mathematical problem*

(a) The right-hand sides of Equations 9 and 10 have been omitted deliberately from the version of the article printed above. Show in detail how the expression for t in Equation 11 is obtained from the formula for $c(t)$ in Equation 8, and in the course of your answer give the missing parts of Equations 9 and 10. [*6*]

(b) It is stated after Equation 11 that this equation 'gives the thickness of insulation for which the total annual cost is a minimum', but nowhere is it shown that the stated value of t does correspond to a minimum (and not a maximum or a point of inflection). Show that the article is correct in asserting that it is a minimum. [*2*]

(v) *Interpret the solution*

(a) One kind of qualitative check that can be made of the solution is to consider whether the formula for t changes in the way one would expect if the values of the parameters are changed. This is something that can be investigated by considering the formula directly, or perhaps by sketching an appropriate graph. As an example, suppose we compare a house in the north of the UK with one in the south. We would expect it to be worthwhile to install more insulation in the north of the country, because winters are harder there than in the south. The model agrees with this expectation, because a larger value of $\Delta\theta$ leads to a larger value of t for minimum $c(t)$. In each of the following cases, consider what happens to the value of t for minimum $c(t)$ according to the model, and comment on the extent to which this result corresponds to your expectations. You should assume in each case that all the other conditions remain unchanged.

A The cost of heating fuels such as gas, electricity and oil goes up.

B There is a rise in the interest rate.

C Heating is required for fewer days during the year.

D The price of insulating material comes down.

E The insulating properties of the material available on the market for domestic loft insulation improve. [7]

(b) Does the model predict any circumstances in which it would make economic sense not to install insulation at all? Explain your answer. [2]

(c) With reference to the numerical calculation in the final paragraph, explain why

A the U value of the ceiling material alone is $16\,\mathrm{W\,m^{-2}\,°C^{-1}}$;

B the cost of $1\,\mathrm{m}^3$ of fibreglass is £17.14;

C it is necessary to include the factor 1000 in the denominator of the fraction under the square root sign. [3]

(vi) *Compare with reality*

It is not possible to compare this model with reality in the sense of comparing its predictions with what would happen in practice, without carrying out some tests of the cost-effectiveness of insulation. But even so, there are some questions that we can ask which help to evaluate how adequate the mathematical model is as a guide to installing loft insulation.

(a) According to the article, the model predicts a layer of insulation thicker than is normally recommended. Does this, in your view, invalidate the model? Give reasons for your answer. [2]

(b) The outcome of the model clearly depends on the assumptions that are made in deriving it. So one important test is to consider how varying those assumptions would affect the results of the model. One assumption which may not always be realistic is that the insulation is laid by the householder. If this assumption is relaxed, it will be necessary to incorporate labour costs in calculating $i(t)$. Make the changes to the model which result from this, explaining clearly how you have taken account of the labour costs, and describe briefly what effects, if any, your changes will have on the outcome of the model. [4]

Question 2

In this question you are asked to carry out some mathematical modelling for yourself. We shall first describe the situation, then make some suggestions about how you should tackle the problem, then give you some data, and finally explain what your tutor will be looking for in your solution.

The situation

One of the major motor-car manufacturers has been showing an intriguing TV advertisement for one of its models recently. This car is fitted with a driver's airbag, and the purpose of the advertisement is to draw the viewer's attention to the benefits of having an airbag. In the advertisement, a car (of the relevant model) is driven very slowly off the flat roof of a tall building. It is seen falling more-or-less vertically downwards towards what appears to be a piece of white material on the ground where it would land. Just before the car is about to smash into the ground, however, the piece of material begins to move: it turns out to be a giant airbag, which inflates fully just in time to arrest the car, which lands on top of the bag at the exact moment when it is fully inflated. The bag then gently subsides, and the car is driven off without having suffered the slightest damage.

Supposing that there was no electronic wizardry involved in the production of the advertisement, and that the events it shows actually happened as they appear to have happened, one cannot help wondering how the film-makers knew when to start blowing up the airbag. It would have been important to get the timing just right. If the inflation of the airbag had been left too late, then the pressure of gas in the bag wouldn't have been sufficient to stop the car hitting the ground. On the other hand, the airbag begins to deflate automatically as soon as it has reached its fullest extent, so as to land the car gently (it behaves just like airbags inside cars in this respect, though they deflate in order not to obscure the windscreen any longer than is necessary). So if inflation had begun too early, the bag would also have been too soft to arrest the car effectively.

You are invited to use your mathematical modelling skills to solve the problem of when to start inflating the airbag.

Method of attack

You should develop a mathematical model of how the car falls which will allow you to predict how far above the ground it is at any given time after it has left the top of the building. You will need to consider whether or not it is necessary to take any horizontal motion of the car into account, in addition to its vertical motion. You will also have to decide whether or not to ignore air resistance, the possibility that the car will rotate as it falls, and any other similar factors. When you have modelled the falling motion of the car, you will have to work out how to use your model to predict when the car must hit the airbag if the 'trick' is to work properly.

Data

The following list should contain all the data you will need to solve the problem – indeed, it should be more than enough.

- The building is 100 metres tall.
- The car is 4.8 metres long, 1.7 metres wide and 1.4 metres high; its mass is 1300 kg.
- The acceleration due to gravity is $9.81\,\mathrm{m\,s^{-2}}$.
- Before it is inflated, the airbag covers an area of $1\,\mathrm{m}^2$; when it is fully inflated, the area it covers is about $25\,\mathrm{m}^2$.
- The time taken to inflate the airbag fully is 1 second.
- The airbag must have been inflating for at least 0.8 seconds for the pressure to have reached a level sufficient to support the car.
- The airbag will remain firm enough to support the car for a tenth of a second after it is fully inflated.
- When the airbag is fully inflated, its top is 3 metres above the ground.

Your solution

You should write a brief account of your solution to this problem. Aim to keep to about the same length as the passage quoted in Question 1. That extract, and the drug therapy and the skid marks reports in *Unit M*, all provide useful examples of what is required. We suggest, however, that for definiteness you should base your solution as far as possible on the summary of Section 4 of *Unit M*, and that in your answer you should make sure you deal with the following points.

(i) *Specify the real problem*

Formulate in your own words a clear statement of the problem to be solved. [*6*]

(ii) *Set up a model*

Define precisely all of the parameters and variables you use, select symbols to represent them, and give appropriate units. (For example: 't is the time since the car left the roof, in seconds'.) Use symbols rather than numerical values for the quantities you need to formulate your model; this will make it easier for you to explain your model, and avoid mistakes. [*6*]

Carefully state the simplifying assumptions which you have made in formulating your model. (For example: 'it is assumed that air resistance may be ignored'.) [*6*]

Based on your assumptions, derive an expression for the downward component of the car's acceleration while it is falling. Explain fully how you have obtained your result. [*3*]

(iii) *Formulate the mathematical problem*

Explain how the knowledge of the downward component of the car's acceleration can be used to predict when it will be necessary to begin inflating the airbag. [*4*]

(iv) *Solve the mathematical problem*

Obtain a formula giving the time at which the inflation of the airbag must begin, in terms of the height of the building, the height of the top of the airbag when it is fully inflated, and any other relevant parameters. [*7*]

(v) *Interpret the solution*

Using the relevant data from the collection given above, obtain a quantitative answer to the initial question; that is, say how long after the car begins to fall inflation of the airbag should be started. [*3*]

Having obtained an answer, you must consider how reliable it is likely to be: if you were doing this for real, rather a large sum of money (the cost of a new car) would hang on the accuracy of your calculations. One obvious cause of possible error is the difficulty of measuring the height of the building sufficiently accurately: for example, the relative error in the height of the building could be 10%. What would be the effect on your answer of a 10% error in the height of the building? [*4*]

Also consider briefly how sensitive your answer is to errors in the other data values. [*3*]

(vi) *Compare with reality*

We certainly do not suggest that you can or should carry out any experiments to test the reliability of your mathematical model. The car in the advertisement takes 14 seconds to fall from the top of the building to the inflated airbag. Do you think that your analysis throws any light on how the advertisement was really produced? Give reasons for your answer. [*4*]

Identify one of the assumptions you made in part (ii) which you think could most profitably be relaxed. Suggest how it could be improved, and say how you think that this will affect your answer. [*4*]

CMA MST204 44 **Cut-off date** 18 June 1998

This assignment covers *Units 15, 17, 18* and *19.*

Unit 15

Questions 1 and 2

A particle of mass 2 kg moves with an acceleration, in $\mathrm{m\,s^{-2}}$, of

$$2t\,\mathbf{i} + \sin \pi t\,\mathbf{j},$$

where $\mathbf{i}$ and $\mathbf{j}$ are Cartesian unit vectors, and t is the time in seconds.

1 What is the velocity of the particle, in $\mathrm{m\,s^{-1}}$, at time $t = 1$, if it is at rest at time $t = 0$?

Options

A $\mathbf{i}$	**B** $2\mathbf{i}$	**C** $2\mathbf{i} + \pi\mathbf{j}$	**D** $2\mathbf{i} - \pi\mathbf{j}$
E $\mathbf{i} + \frac{2}{\pi}\mathbf{j}$	**F** $\mathbf{i} - \frac{2}{\pi}\mathbf{j}$	**G** $\mathbf{i} + \frac{1}{\pi}\mathbf{j}$	**H** $\mathbf{i} - \frac{1}{\pi}\mathbf{j}$

2 What is the magnitude of the force, in newtons, acting on the particle at time $t = 1$?

Options

A $\mathbf{i}$	**B** $2\mathbf{i}$	**C** $4\mathbf{i}$	**D** $4\mathbf{i} + 2\mathbf{j}$
E 1	**F** 2	**G** 4	**H** $2\sqrt{5}$

Questions 3 to 6

A block of mass 2 kg is in contact with a plane surface inclined at an angle of 30° to the horizontal. The coefficient of static friction between the block and the plane is equal to 0.59, whereas the coefficient of kinetic friction is equal to 0.56.

In Questions 3 and 4 it is to be assumed that no forces act on the block other than its weight, the normal reaction and the frictional reaction.

3 Decide whether or not the block can remain at rest on the inclined plane.

Options

A The block can remain at rest on the plane.

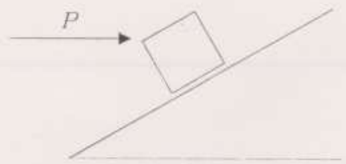
B The block cannot remain at rest on the plane.

4 Select the option which is closest to the magnitude of the frictional reaction, $|\mathbf{F}_f|$, in newtons, if the block is moving *down* the plane. Here g denotes the magnitude of the gravitational acceleration, in $\mathrm{m\,s^{-2}}$.

Options

A $0.48g$	**B** $0.56g$	**C** $0.59g$	**D** $0.87g$
E $0.97g$	**F** $1.00g$	**G** $1.02g$	**H** $1.73g$

An additional force of magnitude P newtons is applied to the block in a horizontal direction when the block is at rest on the inclined plane, as shown in the diagram. This force is such that if it were any stronger, the block would begin to slide *up* the plane.



5 Select the option which gives a correct expression for the magnitude of the normal reaction, $|\mathbf{F}_N|$, in terms of P and g.

6 Select the option which gives a correct expression for the magnitude of the frictional reaction, $|\mathbf{F}_f|$, in terms of P and g.

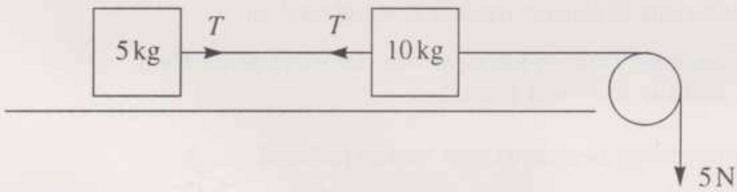
Options for Questions 5 and 6

A g	**B** $\sqrt{3}g$	**C** $P-g$	**D** $-P+g$
E $\frac{1}{2}P+\sqrt{3}g$	**F** $-\frac{1}{2}P+\sqrt{3}g$	**G** $\frac{1}{2}\sqrt{3}P-g$	**H** $-\frac{1}{2}\sqrt{3}P+g$

Unit 17

Questions 7 and 8

Two blocks, of masses 5 kg and 10 kg, are connected by a light inextensible taut string. They move on a frictionless horizontal surface, along a straight line. A further light inextensible string is attached to the heavier block; it extends horizontally along the same straight line to a frictionless pulley, over which it passes. This second string is being pulled vertically downwards by a force of magnitude 5 newtons, as shown in the diagram.



7 What is the magnitude of the common acceleration of the two blocks?

Options

A $\frac{1}{5}\,\mathrm{m\,s^{-2}}$	**B** $\frac{1}{3}\,\mathrm{m\,s^{-2}}$	**C** $\frac{1}{2}\,\mathrm{m\,s^{-2}}$	**D** $1\,\mathrm{m\,s^{-2}}$
E $2\,\mathrm{m\,s^{-2}}$	**F** $3\,\mathrm{m\,s^{-2}}$	**G** $5\,\mathrm{m\,s^{-2}}$	**H** $10\,\mathrm{m\,s^{-2}}$

8 What is the magnitude T of the force exerted by the connecting string on either of the blocks?

Options

A 0 N	**B** $1\frac{1}{3}$ N	**C** $1\frac{2}{3}$ N	**D** 2 N
E $2\frac{1}{2}$ N	**F** 3 N	**G** $3\frac{1}{3}$ N	**H** 5 N

Questions 9 and 10

A lorry is travelling along a straight horizontal road at a constant speed when a car, travelling in the same direction and at a constant but greater speed, comes up behind it and collides with it. As a result of the collision, the two vehicles become locked together.

The mass of the lorry is 4000 kg and its speed before the collision is $15\,\mathrm{m\,s^{-1}}$, while the mass of the car is 1000 kg and its speed before the collision is $20\,\mathrm{m\,s^{-1}}$.

9 What is the speed of the centre of mass of the two vehicles before they collide?

10 What is the common speed of the two vehicles immediately after the collision?

Options for Questions 9 and 10

A $15\,\mathrm{m\,s^{-1}}$	**B** $16\,\mathrm{m\,s^{-1}}$	**C** $16\frac{1}{4}\,\mathrm{m\,s^{-1}}$	**D** $17\,\mathrm{m\,s^{-1}}$
E $17\frac{1}{2}\,\mathrm{m\,s^{-1}}$	**F** $18\,\mathrm{m\,s^{-1}}$	**G** $19\,\mathrm{m\,s^{-1}}$	**H** $20\,\mathrm{m\,s^{-1}}$

Questions 11 and 12

Two particles collide at the origin. Before the collision, particle 1 is moving along the line $y = x$ with speed u, while particle 2, whose mass is twice that of particle 1, is stationary. After the collision, particle 1 moves along the x-axis and particle 2 moves along the y-axis.

11 What is the speed of particle 2 after the collision?

Options

A 0	**B** $\frac{1}{2}u$	**C** u	**D** $2u$
E $\sqrt{2}u$	**F** $2\sqrt{2}u$	**G** $\dfrac{u}{\sqrt{2}}$	**H** $\dfrac{u}{2\sqrt{2}}$

12 Is the collision elastic?

Options

A Yes   **B** No

Unit 18

Questions 13 and 14

Consider the fourth Taylor polynomial about $x = -1$ for the function e^{2x}.

13 Which option gives the coefficient of $(x+1)^4$ in the Taylor polynomial, correct to 3 significant figures?

Options

A 0.00564	**B** 0.0902	**C** 0.667	**D** 4.93
E −0.00564	**F** −0.0902	**G** −0.667	**H** −4.93

14 Select the option which is the error bound given by Taylor's Theorem for the fourth Taylor polynomial about $x = -1$ for e^{2x} in the interval $-2 \le x \le 0$.

Options

A	$\frac{2}{3}\|x+1\|^4$	**B**	$\frac{2}{3}\|x-1\|^4$	**C**	$\frac{4}{15}\|x+1\|^5$
D	$\frac{4}{15}\|x-1\|^5$	**E**	$\frac{2}{3}e^{-4}\|x+1\|^4$	**F**	$\frac{2}{3}e^{-4}\|x+2\|^4$
G	$\frac{4}{15}e^{-4}\|x+1\|^5$	**H**	$\frac{4}{15}e^{-4}\|x+2\|^5$		

Questions 15 and 16

An interpolating quadratic polynomial is used to approximate a function $f(x)$ whose graph passes through the three points $(-1,2)$, $(0,1)$ and $(1,3)$.

15 Select the option which gives the interpolating polynomial.

Options

A	$1+x+2x^2$	**B**	$1-x+3x^2$	**C**	$1-2x+3x^2$
D	$1+\frac{1}{2}x+\frac{3}{2}x^2$	**E**	$1-\frac{1}{2}x+\frac{3}{2}x^2$	**F**	$-1+\frac{1}{2}x+\frac{3}{2}x^2$
G	$-1+\frac{1}{2}x+\frac{7}{2}x^2$	**H**	$-4-\frac{11}{2}x-\frac{3}{2}x^2$		

16 Suppose that $|f^{(3)}(x)| < M$ in the interval $[-1,1]$. Select the option which is the best bound for the error function $\varepsilon(x)$ for the interpolating polynomial over the same interval.

Options

A	$\|\varepsilon(x)\| < \frac{1}{24}\|x(x^2-1)\|M$	**B**	$\|\varepsilon(x)\| < \frac{1}{24}\|(x-1)(x-2)(x-3)\|M$
C	$\|\varepsilon(x)\| < \frac{1}{6}\|x(x^2-1)\|M$	**D**	$\|\varepsilon(x)\| < \frac{1}{6}\|(x-1)(x-2)(x-3)\|M$
E	$\|\varepsilon(x)\| < \frac{1}{2}\|x(x^2-1)\|M$	**F**	$\|\varepsilon(x)\| < \frac{1}{2}\|(x-1)(x-2)(x-3)\|M$
G	$\|\varepsilon(x)\| < \frac{1}{6}\|x^3\|M$	**H**	$\|\varepsilon(x)\| < \frac{1}{2}\|x^3\|M$

Questions 17 and 18

The approximate value of the definite integral

$$\int_0^2 \cos(x^2)\,dx$$

is evaluated numerically by the trapezoidal method and by Simpson's method using subintervals of width $h = 0.5$.

17 Choose the option which gives the value, to four significant figures, obtained by the trapezoidal method.

18 Choose the option which gives the value, to four significant figures, obtained by Simpson's method.

Options for Questions 17 and 18

A	-0.7568	**B**	0.4615	**C**	0.4650	**D**	0.5271
E	0.9300	**F**	1.832	**G**	2.790	**H**	3.665

Question 19

The trapezoidal method is to be used, if possible, to obtain an approximate solution to the differential equation

$$y' = (x + 1)y - x^2$$

with the initial condition

$$y(1) = 1.$$

Select the option which gives the approximate value of $y(1.2)$, correct to 3 decimal places, if the step size is $h = 0.2$, or select Option G if you think that the trapezoidal method cannot be used.

Options

A 0.274 **B** 1.226 **C** 1.398

D 1.622 **E** 1.723 **F** 1.851

G The trapezoidal method, being an implicit method, cannot easily be used for this equation.

Question 20

If the step length h is reduced by a factor of 2 when using the Taylor series method of order 3, by what approximate factor is the local truncation error reduced, assuming that h is small?

Options

A 2 **B** 4 **C** 8 **D** 16

E 32 **F** 64 **G** 128 **H** 256

Question 21

For what positive values of the step size h is the Taylor series method of order 2 guaranteed to be absolutely stable for the differential equation

$$y' = -\frac{10}{x}y + \cot x$$

with the initial condition

$$y(1) = 2,$$

over the interval $1 \leq x \leq 3$? Choose the largest possible interval for h.

Options

A The method is unstable for all values of h.

B $h < 0.2$ **C** $h < 0.4$ **D** $h < 0.6$

E $h < 1.7$ **F** $h < 2.0$ **G** $h < 5.0$

H The method is stable for all values of h.

Questions 22 and 23

A numerical method uses the recurrence relation

$$Y_{r+1} = Y_r + \tfrac{1}{5}h(3Y_r' + 2Y_{r+1}')$$

to find approximate solutions for differential equations of the type

$$y' = m(x, y).$$

22 Select those TWO of the following statements which are TRUE.

Options

A The method is implicit.

B The method is explicit.

C The method is consistent with the differential equation.

D The method is inconsistent with the differential equation.

[*There are TWO correct options.*]

23 Select the option which is the principal term in the local truncation error for this method.

Options

A $\frac{1}{10}h^2y_r''$	**B** $\frac{2}{5}h^2y_r''$	**C** $\frac{1}{2}h^2y_r''$	**D** $\frac{9}{10}h^2y_r''$
E $-\frac{1}{10}h^2y_r''$	**F** $-\frac{2}{5}h^2y_r''$	**G** $-\frac{1}{2}h^2y_r''$	**H** $-\frac{9}{10}h^2y_r''$

Question 24

The following methods can all be used to find numerical approximations for the solution of a certain first-order differential equation, with a given initial condition. The same step size is used in each case, and it may be assumed that with this step size, the method is absolutely stable. Which one of the methods would you expect to give the least accurate approximation?

Options

A Euler's method

B The Taylor series method of order 2

C The Taylor series method of order 3

D The trapezoidal method

E The predictor-corrector method (Euler-trapezoidal)

This assignment covers *Units 20, 21* and *22*.

Unit 20

Questions 1 to 6

Consider the matrices

$$\mathbf{P} = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 5 & 2 \\ -2 & 3 \end{bmatrix},$$

$$\mathbf{S} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 3 & 4 \end{bmatrix}.$$

1 Which one of the following options is equal to $2(\mathbf{P} - \mathbf{R}) + 3\mathbf{Q}$?

Options

A $\begin{bmatrix} -9 & 5 \\ 4 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 11 & 5 \\ 10 & -5 \end{bmatrix}$ **C** $\begin{bmatrix} -9 & 5 \\ 10 & -5 \end{bmatrix}$

D $\begin{bmatrix} 11 & 5 \\ 10 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 5 & 13 \\ -2 & 11 \end{bmatrix}$ **F** $\begin{bmatrix} 14 & 14 \\ -2 & 7 \end{bmatrix}$

2 Which one of the following options is equal to $\mathbf{Q}^{-1}$?

Options

A $\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$ **B** $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$ **C** $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$

D $\begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ **F** $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

3 Which one of the following options is equal to $\mathbf{Q}^T\mathbf{R}$?

Options

A $\begin{bmatrix} 9 & 8 \\ 3 & 5 \end{bmatrix}$ **B** $\begin{bmatrix} 9 & 3 \\ 4 & -5 \end{bmatrix}$ **C** $\begin{bmatrix} 5 & 4 \\ 7 & -1 \end{bmatrix}$

D $\begin{bmatrix} 5 & 4 \\ -2 & -3 \end{bmatrix}$ **E** $\begin{bmatrix} 5 & 2 \\ -4 & -3 \end{bmatrix}$ **F** $\begin{bmatrix} 1 & 8 \\ 7 & -1 \end{bmatrix}$

4 Which TWO of the following options represent matrix products which cannot be formed?

Options

A $\mathbf{PS}$ **B** $\mathbf{Qy}$ **C** $\mathbf{Ry}^T$ **D** $\mathbf{x}^T\mathbf{P}^{-1}$

E $\mathbf{yS}$ **F** $\mathbf{xy}$ **G** $\mathbf{yx}$ **H** $\mathbf{SQ}$

[*There are TWO correct options.*]

5 Which one of the following options is the value of $\det(\mathbf{PQ})$?

Options

A -6 **B** -4 **C** -2 **D** 0

E 1 **F** 2 **G** 4 **H** 6

6 There are two values of the number k for which the matrix $\mathbf{Q} + \frac{1}{2}\mathbf{R} + k\mathbf{P}$ is singular. Select the option which gives both values to three decimal places, or select Option H if you think none of the others is correct.

Options

A	-0.667 and -0.250	**B**	-0.667 and 3.500	**C**	-0.250 and 3.500
D	-0.075 and 1.675	**E**	0.075 and 1.675	**F**	0.250 and 0.667
G	0.250 and 3.500	**H**	None of the other options is correct.		

Questions 7 to 9

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

and its inverse $\mathbf{B} = \mathbf{A}^{-1}$.

7 What is the value of $\det \mathbf{A}$?

Options

A	-4	**B**	-2	**C**	0	**D**	1
E	2	**F**	3	**G**	4	**H**	5

8 Select the option which gives the value of the element b_{13} of the matrix $\mathbf{B}$.

9 Select the option which gives the value of the element b_{22} of the matrix $\mathbf{B}$.

Options for Questions 8 and 9

A	-2	**B**	-1	**C**	$-\frac{1}{2}$	**D**	0
E	$\frac{1}{2}$	**F**	1	**G**	2		

Questions 10 and 11

New coordinate axes are defined in terms of given coordinate axes Oxy as follows: the new origin O' is at the point $(-2, 1)$ with respect to the axes Oxy, and the new axes $O'x'y'$ are at a clockwise angle of $30°$ with respect to the axes Oxy.

10 Choose the option which gives the coordinates with respect to $O'x'y'$ of the point whose coordinates with respect to Oxy are $(-1, -3)$.

11 Choose the option which gives the coordinates with respect to Oxy of the point whose coordinates with respect to $O'x'y'$ are $(1, -4)$.

Options for Questions 11 and 12

A	$\left(-2\sqrt{3} + \frac{1}{2}, \frac{1}{2}\sqrt{3} - 4\right)$	**B**	$\left(-2\sqrt{3} + \frac{1}{2}, \frac{1}{2}\sqrt{3} + 2\right)$
C	$\left(-\frac{1}{2}\sqrt{3} + \frac{5}{2}, -\frac{5}{2}\sqrt{3} + \frac{3}{2}\right)$	**D**	$\left(\frac{1}{2}\sqrt{3} - 4, -2\sqrt{3} + \frac{1}{2}\right)$
E	$\left(\frac{1}{2}\sqrt{3} - 2, -2\sqrt{3} - \frac{1}{2}\right)$	**F**	$\left(\frac{1}{2}\sqrt{3}, -2\sqrt{3} + \frac{3}{2}\right)$
G	$\left(\frac{1}{2}\sqrt{3} + 2, -2\sqrt{3} + \frac{1}{2}\right)$	**H**	$\left(\frac{3}{2}\sqrt{3} + \frac{5}{2}, -\frac{5}{2}\sqrt{3} + \frac{3}{2}\right)$

Question 12

The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are square and of the same size, and $\mathbf{A}$ is non-singular; $\mathbf{I}$ is the unit matrix of the same size. Which TWO of the following statements are NOT necessarily true?

Options

A If $\mathbf{AB} = \mathbf{0}$, then $\mathbf{B} = \mathbf{0}$.

B If $\mathbf{B}^2 = \mathbf{0}$, then $\mathbf{B} = \mathbf{0}$.

C If $\mathbf{B}^2 = \mathbf{0}$, then $\mathbf{I} + \mathbf{B}$ is non-singular and $(\mathbf{I} + \mathbf{B})^{-1} = \mathbf{I} - \mathbf{B}$.

D For all $\mathbf{C}$, $(\mathbf{I} + \mathbf{C})^2 = \mathbf{I} + 2\mathbf{C} + \mathbf{C}^2$.

E For all $\mathbf{B}$ and $\mathbf{C}$, $(\mathbf{B} + \mathbf{C})(\mathbf{B} - \mathbf{C}) = \mathbf{B}^2 - \mathbf{C}^2$.

F For all $\mathbf{B}$, $(\mathbf{B}^T)^T = \mathbf{B}$.

G For all non-singular $\mathbf{A}$, $\mathbf{A}^T$ is non-singular.

H For all $\mathbf{B}$ and $\mathbf{C}$, $\det(\mathbf{B}^T\mathbf{C}^T) = \det(\mathbf{BC})$.

[*There are TWO correct options.*]

Unit 21

Question 13

A certain 2×2 matrix $\mathbf{P}$ has eigenvalues -2 and 3. Which option gives the eigenvalues of $\mathbf{P}^2 + 2\mathbf{P} - \mathbf{I}$?

Options

A -9 and 14	B -8 and 15	C -7 and 14	D -7 and 16
E -1 and 14	F -1 and 16	G 0 and 15	H 1 and 16

Questions 14 to 17

Consider the matrix

$$\begin{bmatrix} 5 & -8 \\ 4 & -7 \end{bmatrix}.$$

14 The direct iteration method for finding eigenvalues is applied to this matrix, with initial vector $\mathbf{y}_0 = [0 \quad 1]^T$. Which option gives the scaled vector $\mathbf{y}_1$?

Options

A $[-8 \quad -7]^T$ B $[8 \quad 7]^T$ C $[-\frac{8}{7} \quad -1]^T$

D $[\frac{8}{7} \quad 1]^T$ E $[-1 \quad -\frac{7}{8}]^T$ F $[1 \quad \frac{7}{8}]^T$

G $\left[-\frac{8}{\sqrt{113}} \quad -\frac{7}{\sqrt{113}}\right]^T$ H $\left[\frac{8}{\sqrt{113}} \quad \frac{7}{\sqrt{113}}\right]^T$

15 The inverse iteration method with $p = 0$ for finding eigenvalues is applied to this matrix with initial vector $\mathbf{y}_0 = [0 \quad 1]^T$. Which option gives the scaled vector $\mathbf{y}_1$?

Options

A $[8 \quad 5]^T$	B $[1 \quad \frac{5}{8}]^T$	C $[\frac{8}{5} \quad 1]^T$	D $[-\frac{8}{3} \quad -\frac{5}{3}]^T$
E $[-4 \quad 5]^T$	F $[1 \quad -\frac{5}{4}]^T$	G $[-\frac{4}{5} \quad 1]^T$	H $[\frac{4}{3} \quad -\frac{5}{3}]^T$

16 The LR method for finding eigenvalues is applied to this matrix. Which option gives the matrix $\mathbf{L}_0$?

Options

A $\begin{bmatrix} 1 & 0 \\ \frac{4}{5} & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 0 \\ \frac{5}{4} & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

D $\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 5 & -8 \\ 0 & -\frac{3}{5} \end{bmatrix}$ **F** $\begin{bmatrix} 5 & 0 \\ 4 & -7 \end{bmatrix}$

17 Which option gives the matrix $\mathbf{U}_0$ when the LR method is applied to the given matrix?

Options

A $\begin{bmatrix} 5 & -8 \\ 0 & -\frac{5}{3} \end{bmatrix}$ **B** $\begin{bmatrix} 5 & -8 \\ 0 & -\frac{3}{5} \end{bmatrix}$ **C** $\begin{bmatrix} 5 & -8 \\ 0 & -\frac{3}{7} \end{bmatrix}$

D $\begin{bmatrix} 5 & -8 \\ 0 & \frac{3}{7} \end{bmatrix}$ **E** $\begin{bmatrix} 5 & -8 \\ 0 & \frac{3}{5} \end{bmatrix}$ **F** $\begin{bmatrix} 5 & -8 \\ 0 & \frac{5}{3} \end{bmatrix}$

Question 18

The matrix

$$\begin{bmatrix} 1 & 4 & 9 & 16 \\ 4 & 1 & 4 & 9 \\ 9 & 4 & 1 & 4 \\ 16 & 9 & 4 & 1 \end{bmatrix}$$

has eigenvalues approximately equal to -16.81, -3.32, -1.19 and 25.32. Which of the following options gives a value of p which could be used in the inverse iteration method in order to find a more accurate value for the eigenvalue near -3.32?

Options

A -10 **B** -2 **C** 0 **D** 2 **E** 10

Unit 22

Question 19

The matrix $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ has an eigenvalue $\lambda = 1$ with a corresponding eigenvector $[1 \quad -1]^T$, and an eigenvalue $\lambda = 5$ with a corresponding eigenvector $[1 \quad 1]^T$.

Select the option which is the general solution of the system of differential equations

$$\dot{x}_1 = 3x_1 + 2x_2,$$
$$\dot{x}_2 = 2x_1 + 3x_2.$$

Options

A $x_1 = ae^{3t}$, $x_2 = be^{3t}$

B $x_1 = ae^{3t}$, $x_2 = be^{2t}$

C $x_1 = ae^{t} + be^{5t}$, $x_2 = -ae^{t} + be^{5t}$

D $x_1 = ae^{t} - be^{5t}$, $x_2 = ae^{t} + be^{5t}$

E $x_1 = 3ae^{t} + 2be^{5t}$, $x_2 = 2ae^{t} + 3be^{5t}$

F $x_1 = ae^{3t} + be^{2t}$, $x_2 = ae^{2t} + be^{3t}$

G $x_1 = ae^{3t} + be^{2t}$, $x_2 = -ae^{3t} + be^{2t}$

Questions 20 and 21

The matrix

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

has an eigenvalue $\lambda = 0$ with corresponding eigenvector $[1 \quad 1]^T$, and an eigenvalue $\lambda = 4$ with corresponding eigenvector $[1 \quad -1]^T$.

20 Select the option which is the general solution of the system of differential equations

$$\dot{x}_1 = 2x_1 - 2x_2,$$
$$\dot{x}_2 = -2x_1 + 2x_2.$$

Options

A $x_1 = ae^{4t}$, $x_2 = ae^{4t}$

B $x_1 = ae^{4t}$, $x_2 = -ae^{4t}$

C $x_1 = ae^{2t} + be^{-2t}$, $x_2 = ae^{2t} - be^{-2t}$

D $x_1 = ae^{2t} + be^{-2t}$, $x_2 = -ae^{2t} + be^{-2t}$

E $x_1 = at + be^{4t}$, $x_2 = at - be^{4t}$

F $x_1 = at + be^{4t}$, $x_2 = -at + be^{4t}$

G $x_1 = a + be^{4t}$, $x_2 = a - be^{4t}$

H $x_1 = a + be^{4t}$, $x_2 = -a + be^{4t}$

21 Select the option which is the general solution of the system of differential equations

$$\ddot{x}_1 = 2x_1 - 2x_2,$$
$$\ddot{x}_2 = -2x_1 + 2x_2.$$

In the options, C_1, C_2, C_3 and C_4 are arbitrary constants.

Options

A $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t}$

B $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} (C_1 e^{2t} + C_2 e^{-2t})$

C $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} (C_1 \cos 2t + C_2 \sin 2t)$

D $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (C_2 e^{2t} + C_3 e^{-2t})$

E $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (C_2 \cos 2t + C_3 \sin 2t)$

F $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} (C_1 + C_2 t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (C_3 e^{2t} + C_4 e^{-2t})$

G $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} (C_1 + C_2 t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (C_3 \cos 2t + C_4 \sin 2t)$

Question 22

The matrix

$$\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

has an eigenvalue $\lambda = -1 + i$ with a corresponding eigenvector $[1 \quad i]^T$, and an eigenvalue $\lambda = -1 - i$ with a corresponding eigenvector $[1 \quad -i]^T$.

Select the option which is the general solution of the system of differential equations

$$\dot{x}_1 = -x_1 + x_2,$$
$$\dot{x}_2 = -x_1 - x_2.$$

Options

A $x_1 = Ae^t \cos t + Be^t \sin t$
$x_2 = Ae^t \sin t + Be^t \cos t$

B $x_1 = Ae^t \cos t + Be^t \sin t$
$x_2 = Ae^t \sin t - Be^t \cos t$

C $x_1 = Ae^t \cos t + Be^t \sin t$
$x_2 = -Ae^t \sin t + Be^t \cos t$

D $x_1 = Ae^t \cos t + Be^t \sin t$
$x_2 = -Ae^t \sin t - Be^t \cos t$

E $x_1 = Ae^{-t} \cos t + Be^{-t} \sin t$
$x_2 = Ae^{-t} \sin t + Be^{-t} \cos t$

F $x_1 = Ae^{-t} \cos t + Be^{-t} \sin t$
$x_2 = Ae^{-t} \sin t - Be^{-t} \cos t$

G $x_1 = Ae^{-t} \cos t + Be^{-t} \sin t$
$x_2 = -Ae^{-t} \sin t + Be^{-t} \cos t$

H $x_1 = Ae^{-t} \cos t + Be^{-t} \sin t$
$x_2 = -Ae^{-t} \sin t - Be^{-t} \cos t$

Questions 23 and 24

The matrix

$$\mathbf{A} = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix}$$

has an eigenvalue $\lambda = -1$ with corresponding eigenvector $[2 \quad -1]^T$, and an eigenvalue $\lambda = -4$ with corresponding eigenvector $[1 \quad 1]^T$. A particle moves in a plane in such a way that its coordinates (x, y) satisfy

$$\ddot{\mathbf{r}} = \mathbf{A}\mathbf{r},$$

where $\mathbf{r} = [x \quad y]^T$.

23 Along which TWO straight lines can the particle move in simple harmonic motion?

Options

A $y = 2x$ **B** $y = x$ **C** $y = \frac{1}{2}x$

D $y = -2x$ **E** $y = -x$ **F** $y = -\frac{1}{2}x$

[*There are TWO correct options.*]

24 Select the option which gives the general form for the position of the particle, at time t, for any initial conditions. In the options, C_1, C_2, C_3 and C_4 are arbitrary constants.

Options

A $x = 2C_1 \cos t + 2C_2 \cos 2t + C_3 \sin t + C_4 \sin 2t$
$y = -C_1 \cos t - C_2 \cos 2t + C_3 \sin t + C_4 \sin 2t$

B $x = 2C_1 \cos t + 2C_2 \sin t + C_3 \cos 4t + C_4 \sin 4t$
$y = -C_1 \cos t - C_2 \sin t + C_3 \cos 4t + C_4 \sin 4t$

C $x = 2C_1 \cos t + 2C_2 \sin t + C_3 \cos 2t + C_4 \sin 2t$
$y = -C_1 \cos t - C_2 \sin t + C_3 \cos 2t + C_4 \sin 2t$

D $x = 2C_1 \cos t + 2C_2 \sin t + C_3 \cos 2t + C_4 \sin 2t$
$y = -C_1 \cos t - C_2 \sin t - C_3 \cos 2t - C_4 \sin 2t$

E $x = C_1 \cos t + C_2 \sin t + 2C_3 \cos 2t + 2C_4 \sin 2t$
$y = C_1 \cos t + C_2 \sin t - C_3 \cos 2t - C_4 \sin 2t$

This assignment covers *Units 17, 18, 19, 21* and *22*.

There are five questions in this assignment; Question 3 and Question 4 are both allotted fewer than 25 marks, however, and the total number of marks for the assignment is 100.

Note that there is no tutor-marked assignment question on *Unit 20*; this unit is assessed by computer-marked assignment questions only.

Question 1 (*Unit 17*)

Note that the total number of marks available for this question is 25.

A toy rocket is initially at rest on its launch pad and points vertically upwards, the motors being switched on at time $t = 0$. The total initial mass of the rocket (including its fuel) is 2 kg. The fuel is burnt in such a way that the mass m of the rocket at time t satisfies

$$\frac{dm}{dt} = -\tfrac{1}{50}t \qquad (0 \le t \le 10).$$

The rocket contains sufficient fuel for the motors to run for 10 seconds, and the exhaust gases are ejected vertically downwards at a constant speed of $490\,\mathrm{m\,s^{-1}}$ relative to the rocket casing. For the motion of the rocket you may neglect any effects due to air resistance and assume that the magnitude g of the acceleration due to gravity is equal to $10\,\mathrm{m\,s^{-2}}$. Throughout the question all the variables are measured in the appropriate SI units.

(i) Find an expression for the mass $m(t)$ of the rocket for the period $0 \le t \le 10$ when its motors are running. [*3*]

(ii) Draw a diagram showing the external forces acting on the rocket during the period in which the rocket remains on the launch pad with the motors on. [*2*]

(iii) Find the upward reaction force exerted by the launch pad on the rocket at time t during the period in which the rocket remains on the launch pad with the motors on. [*6*]

(iv) Show that the rocket leaves the launch pad 2 seconds after the motors are switched on. [*3*]

(v) Draw a diagram showing the external forces acting on the rocket during its vertical flight. [*1*]

(vi) Show that during its flight, the speed v of the rocket satisfies the differential equation

$$\frac{dv}{dt} = \frac{980t}{200 - t^2} - 10 \qquad (2 \le t \le 10).$$ [*3*]

(vii) Find the speed of the rocket at the instant when the fuel is just exhausted, expressing your answer to an accuracy of one decimal place. [*7*]

Question 2 (*Unit 18*)

Note that the total number of marks available for this question is 25.

(a) (i) Show that the quadratic polynomial $p(x)$ which interpolates the function

$$f(x) = \cos x$$

at the points $x = 0$, 0.5 and 1 is

$$p(x) = 1 - 0.029\,97x - 0.429\,73x^2,$$

where the coefficients are correct to 5 decimal places. [*6*]

(ii) The equation

$$\cos x = 3x$$

has a single root, which is near 0.3. By using $p(x)$ as an approximation to $\cos x$, and solving a quadratic equation, find an approximation to this root better than 0.3, giving your answer to 3 decimal places. [7]

(iii) Use the Newton–Raphson method to determine the root of the equation $\cos x = 3x$ to an accuracy of 6 decimal places. [7]

(b) Consider the differential equation

$$\frac{dy}{dx} = x^2 - y^2 \qquad (x \geq 0)$$

with initial condition $y(1) = 0$. Use the Taylor series method of order 2 with step size $h = 0.1$ to find an approximation to $y(1.2)$, where y is the solution to the equation with the given initial condition. Give your answer correct to 6 decimal places. [5]

Question 3 (*Unit 19*)

Note that the total number of marks available for this question is 13.

This question is concerned, once again, with the solution of the differential equation

$$\frac{dy}{dx} = x^2 - y^2$$

with the initial condition $y(1) = 0$.

(i) Starting from the general form of the recurrence relations for the predictor-corrector (Euler-trapezoidal) method, write down the recurrence relations for finding a numerical solution to the above differential equation using this method. Use your calculator to find an approximation to $y(1.2)$, where y is the solution satisfying the given initial condition, using a step size $h = 0.1$. Give your answer correct to 6 decimal places. [5]

(ii) The values shown in the table below were obtained for the approximations to $y(2)$ given by the predictor-corrector (Euler-trapezoidal) method with various step lengths h by using a computer. Plot a graph of these approximations to $y(2)$ against h^2.

Step length h	Approximation to $y(2)$
0.05	1.565 785
0.1	1.564 497
0.125	1.563 497
0.2	1.559 028
0.25	1.554 916

What does your graph indicate to be the relationship between the predictor-corrector (Euler-trapezoidal) approximations to $y(2)$ and the step length h for small h? Use your graph to estimate the true value of $y(2)$ correct to four decimal places. [4]

(iii) By using one of the entries in the above table and the estimate for the true value of $y(2)$ you obtained in part (ii), calculate the maximum value of the step length h which you would have to use in the predictor-corrector (Euler-trapezoidal) method to ensure an accuracy of six decimal places in the approximation to $y(2)$. [4]

Question 4 (*Unit 21*)

Note that the total number of marks available for this question is 12.

Find the eigenvalues and corresponding eigenvectors of the matrix

$$\begin{bmatrix} 1 & 4 & -1 \\ -1 & 0 & -1 \\ -2 & -4 & 0 \end{bmatrix}.$$ [12]

Question 5 (*Unit 22*)

Note that the total number of marks available for this question is 25.

This question is concerned with the system of differential equations

$$\dot{x}_1 = 7x_1 - 3x_2 - e^{2t},$$
$$\dot{x}_2 = 6x_1 - 2x_2.$$

(i) Express the system of differential equations in the form

$$\dot{\mathbf{x}}(t) = \mathbf{B}\mathbf{x}(t) + \mathbf{h}(t),$$

where

$$\mathbf{x}(t) = [x_1(t) \quad x_2(t)]^T, \qquad \mathbf{h}(t) = [h_1(t) \quad h_2(t)]^T$$

and $\mathbf{B}$ is a constant 2×2 matrix. [2]

(ii) Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{B}$. Hence find a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ is a diagonal matrix. [8]

(iii) Find the general solution of the system of differential equations. [15]

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